

# Holographic electrical and thermal conductivity in strongly coupled gauge theory with multiple chemical potentials

Sachin Jain<sup>1</sup>

*Institute of Physics,  
Bhubaneswar 751 005, India.*

## ABSTRACT

We study transport coefficients of strongly coupled gauge theory in the presence of multiple chemical potential which are dual to rotating D3, M2 and M5 brane. Using the general form of the perturbation equations, we compute electrical conductivity at finite temperature as well as at zero temperature. We also study thermal conductivity for the same class of black holes and show that thermal conductivity and viscosity obeys Wiedemann-Franz like law even in the presence of multiple chemical potential.

Dec 2009

---

<sup>1</sup>e-mail: sachjain@iopb.res.in

# 1 Introduction

One of the hallmark of gauge/gravity duality is to allow us to gain insights into strongly coupled gauge theories both at zero and non-zero temperature. Even though these theories, in several ways, different from realistic gauge theories like QCD, they do share qualitatively similar behavior in many respects. For example, in large  $N_c$  limit, and in the presence of an infrared cutoff, these theories de-confine at high temperature. Viscosity to entropy ratio of these theories are quite low as suggested by RHIC data. In all these cases, one could carry out trustable calculations using conjectured gauge-gravity duality. It allows us to map a strongly coupled gauge theories to a weakly coupled gravity. In particular, in the large  $N_c$  and large t'Hooft coupling limit, the dual description turns out to be various black holes in anti deSitter (AdS) space.

In the above framework the low energy or long wavelength behavior of  $\mathcal{N} = 4$  supersymmetric  $SU(N_c)$  Yang-Mills was studied in [1], in the presence of non-zero chemical potential. Generically, three independent chemical potentials can be associated for three  $U(1)$  charges in the Cartan sub-algebra of  $SO(6)_R$ . The dual then represents a five dimensional R-charged black hole as discussed in [2]. Though shear viscosity was calculated in the presence of three chemical potentials, thermal conductivity was calculated only when a single chemical potential is non-zero. It was found that an appropriate ratio of thermal conductivity and the shear viscosity is  $8\pi^2$ . This is the analogous Wiedemann-Franz law for this gauge theory [1]. Subsequently there were many other works [3] (see [4] and references therein) along these directions. In a recent paper [5], some of the above results were generalized in the presence of multiple chemical potential. Besides providing a general framework and solutions of certain long wavelength perturbation of the geometry, we computed diffusion coefficients associated with these black holes carrying multiple R-charges in four, five and seven dimensions. We found that near the second order unstable points of these holes, the diffusion coefficient is zero - a fact consistent with the critical slowing down [6] .

It turns out that, regardless of the detail structure of these black holes, many of the above calculations for electrical conductivity can be simplified and can be given a general structure. This is what we first attempt to do in this paper. We then use these results to compute thermal conductivities in the presence of *multiple* R-charges in *various* dimensions. Furthermore, we show that the thermal conductivity and viscosity ratio remains same and is equal to their single charge cosines. Subsequently, we turn our attention to the extremal R-charged black holes. Recently, it was understood that studying perturbations of these black holes requires special care [7]. The approach was then used to compute viscosity to entropy for a rather large class of extremal black holes [8]. In this paper, we also compute the conductivity of gauge theory dual to multiple R-charged extremal black hole and give a general expression for the dimension of dual

operators. We then find an effective schrodinger problem which sheds light to some of the results obtained in the text.

This paper is structured as follows. In the next section, we give a general framework to study relevant perturbation equations in the gravity side. This simplifies the computations of previous works. In section 3, we use effective action approach to give a general expression of the electrical conductivity matrix in a way which is insensitive to many details of the explicit geometry. In section 4, we carry out an analysis and discussion on conductivity for black holes with different charges individually. In the later section, we turn our attention on the computation of thermal conductivity for multiple R-charged black holes and analyze Wiedemann-Franz law. We find that in five, four and seven dimensions the appropriate ratio of thermal conductivity and viscosity, regardless of the number of charge contents, are  $8\pi^2$ ,  $32\pi^2$  and  $2\pi^2$  respectively. In section 6, we study the electrical conductivity of the extremal R-charged black holes. In the next section, we map the problem to Schrödinger problem and find out behavior of effective Schrödinger potential for both extremal as well as non-extremal case. Last section of this paper contains a discussion of our result. At the end we collect some earlier results which are used in the main body of the paper.

## 2 General perturbation equation

In this section we write down perturbation equation required for computation of electrical conductivity. For that we consider a lagrangian of the form

$$\frac{\mathcal{L}}{\sqrt{-g}} = R - \frac{1}{4}G_{ij}F_{\mu\nu}^i F^{\mu\nu j} + \text{Scalar field terms} + \text{Chern - Simons term}, \quad (2.1)$$

where  $F^{\mu\nu j}$  is the field-strength tensor of the  $j$ -th  $U(1)$  gauge field. The metric that we take is of the form

$$ds^2 = g_{tt}dt^2 + g_{uu}du^2 + g_{xx} \sum_{i=1}^{d-1} (dx^i)^2. \quad (2.2)$$

As an example one can consider R-charged black holes in various dimensions (see [1], [2]).

Since our aim is to compute the electrical conductivity using Kubo formula, it is sufficient to consider perturbations in the tensor (metric) and the vector (gauge fields) modes around the black hole solution and keep other fields such as scalars unperturbed. So perturbations are of the form:

$$g_{\mu\nu} = \mathbf{g}_{\mu\nu}^{(0)} + h_{\mu\nu}, \quad A_{\mu}^i = \mathbf{A}_{\mu}^{i(0)} + \mathcal{A}_{\mu}^i, \quad i = 1, 2 \quad . \quad (2.3)$$

where  $\mathbf{g}_{\mu\nu}^{(0)}$  and  $\mathbf{A}_{\mu}^{i(0)}$  are background metric and gauge fields. In order to determine R-charge conductivity it is enough to consider perturbations in  $(tx)$  and  $(xx)$  component

of the metric tensor and  $x$  component of the gauge fields. Moreover one can choose the perturbations to depend on radial coordinate  $u$ , time  $t$  and one of the spatial worldvolume coordinate  $z$ . A convenient ansatz with the above restrictions in mind is

$$h_{tx} = \mathbf{g}_{0xx} T(u) e^{-i\omega t + iqz}, \quad h_{zx} = \mathbf{g}_{0xx} Z(u) e^{-i\omega t + iqz}, \quad \mathcal{A}_x^i = \frac{\mu^i}{2} \Phi_i(u) e^{-i\omega t + iqz}. \quad (2.4)$$

Here  $\omega$  and  $q$  represent the frequency and momentum in  $z$  direction respectively and we set perturbations in the other components to be equal to zero. Also note that,  $\mu^i$  represents the chemical potential for the  $i^{th}$  gauge field. Next step is to find linearized equations which follows from the equations of motion. It turns out that at the level of linearized equation and at zero momentum limit metric perturbation  $Z(u)$  decouple from the rest. One can further eliminate  $T(u)$  reducing it to equation for perturbations in gauge fields only. After substitution one finds the equations for perturbed gauge fields to be

$$\begin{aligned} \frac{d^2}{du^2} \Phi_i(u) &+ \left( \frac{d}{du} \log(\sqrt{-g} G_{ii} g^{xx} g^{uu}) \right) \frac{d}{du} \Phi_i(u) - \omega^2 \frac{g_{uu}}{g_{tt}} \Phi_i(u) \\ &+ \frac{1}{g_{tt}} \frac{F_{ut}^i}{\mu^i} \left( \sum_{j=1}^m \mu^j G_{jj} F_{ut}^j \Phi_j(u) \right) = 0. \end{aligned} \quad (2.5)$$

The last term in equation (2.5) comes from the substitution of the metric perturbation part which can be looked upon as interaction between different gauge fields. Note that all the equation written in [5], [6] can be reproduced from the equation (2.5). One can rewrite equation (2.5) as

$$\frac{d}{du} \left( N_i \frac{d}{du} \Phi_i(u) \right) - \omega^2 N_i g_{uu} g^{tt} \Phi_i(u) + \sum_{j=1}^m M_{ij}^1 \Phi_j(u) = 0, \quad (2.6)$$

where

$$N_i = \sqrt{-g} G_{ii} g^{xx} g^{uu}. \quad (2.7)$$

and

$$M_{ij}^1 = \frac{F_{ut}^i}{\mu^i} \sqrt{-g} G_{ii} g^{xx} g^{uu} g^{tt} \mu^j G_{jj} F_{ut}^j. \quad (2.8)$$

Note that  $M_{ij}^1 \neq M_{ji}^1$ . As an example consider R-charge black holes in 5-dimension (with 3 R-charges, see [5]). For this case we get

$$\begin{aligned} (\Phi_i)'' &+ \left( \frac{f'}{f} + 2 \frac{H'_i}{H_i} - \frac{\mathcal{H}'}{\mathcal{H}} \right) (\Phi_i)' + \frac{(\frac{\omega}{2\pi T_0})^2 \mathcal{H}}{f^2 u} \Phi_i - \frac{u(1+k_i)}{f H_i^2} \left[ k_1(1+k_2)(1+k_3) \Phi_1 \right. \\ &\left. + k_2(1+k_1)(1+k_3) \Phi_2 + k_3(1+k_1)(1+k_2) \Phi_3 \right] = 0. \end{aligned} \quad (2.9)$$

It is convenient to redefine  $\Phi$  as  $\phi_j(u) = \mu^j \Phi_j(u)$ .<sup>1</sup> In the redefined variables equa-

---

<sup>1</sup>No summation. Unless explicitly written there is no summation for repeated indices in this paper.

tions are

$$\frac{d}{du}(N_i \frac{d}{du} \phi_i(u)) - \omega^2 N_i g_{uu} g^{tt} \phi_i(u) + \sum_{j=1}^m M_{ij} \phi_j(u) = 0. \quad (2.10)$$

with

$$N_i = \sqrt{-g} G_{ii} g^{xx} g^{uu}. \quad (2.11)$$

and

$$M_{ij} = F_{ut}^i \sqrt{-g} G_{ii} g^{xx} g^{uu} g^{tt} G_{jj} F_{ut}^j. \quad (2.12)$$

Note that  $M_{ij} = M_{ji}$ . For evaluating conductivity in the low frequency limit for non-extremal backgrounds we only need to solve equations up to zeroth order in  $\omega$ . To that order one finds,

$$\frac{d}{du}(N_i \frac{d}{du} \phi_i(u)) + \sum_{j=1}^m M_{ij} \phi_j(u) = 0. \quad (2.13)$$

### 3 Effective action and expression for conductivity

In this section, after writing down effective action (which reproduces (2.13)), we extract out the expression for electrical conductivity using Kubo formula. Effective action (for discussion on effective action for single charge case see [9]) can be written as

$$\begin{aligned} S = & \frac{1}{16\pi G_{d+1}} \int \frac{d^d q}{(2\pi)^d} du \left[ \sum_{i=1}^m N_i(u) \frac{d}{du} \phi_i(u, \omega, q) \frac{d}{du} \phi_i(u, -\omega, -q) \right. \\ & \left. + \sum_{i,j=1}^m M_{ij}(u) \phi_i(u, \omega, q) \phi_j(u, -\omega, -q) \right]. \end{aligned} \quad (3.1)$$

Boundary action is given by

$$S_\epsilon = \lim_{u \rightarrow 0} \frac{1}{16\pi G_{d+1}} \int \frac{d^d q}{(2\pi)^d} \left( \sum_{i=1}^m N_i(u) \frac{d}{du} \phi_i(u, \omega, q) \phi_i(u, -\omega, -q) \right). \quad (3.2)$$

For calculating conductivity we need to look for

$$\Im \left[ \lim_{u \rightarrow 0} \frac{1}{16\pi G_{d+1}} \int \frac{d^d q}{(2\pi)^d} \left( \sum_{i=1}^m N_i(u) \frac{d}{du} \phi_i(u, \omega, q) \phi_i(u, -\omega, -q) \right) \right] \quad (3.3)$$

Now

$$\begin{aligned} & \frac{d}{du} \Im \left( \sum_{i=1}^m N_i(u) \frac{d}{du} \phi_i(u, \omega, q) \phi_i(u, -\omega, -q) \right) \\ &= \Im \left[ \sum_{i=1}^m \frac{d}{du} \left( N_i(u) \frac{d}{du} \phi_i(u, \omega, q) \right) \phi_i(u, -\omega, -q) \right. \\ & \quad \left. + \sum_{i=1}^m N_i(u) \frac{d}{du} \phi_i(u, \omega, q) \frac{d}{du} \phi_i(u, -\omega, -q) \right]. \end{aligned} \quad (3.4)$$

Using (2.13) r.h.s of above equation reduces to

$$\Im \left[ - \sum_{i,j=1}^m M_{ij}(u) \phi_i(u, \omega, q) \phi_j(u, -\omega, -q) + \sum_{i=1}^m N_i(u) \frac{d}{du} \phi_i(u, \omega, q) \frac{d}{du} \phi_i(u, -\omega, -q) \right], \quad (3.5)$$

which is equal to zero since quantity in the bracket is real. Then (3.3) can as well be calculated at the horizon i.e. at  $u = 1$ , which simplifies calculations significantly. Regularity at the horizon implies

$$\lim_{u \rightarrow 1} \frac{d}{du} \phi_i(u) = -i\omega \lim_{u \rightarrow 1} \sqrt{-\frac{g_{uu}}{g_{tt}}} \phi_i(u) + \mathcal{O}(\omega^2). \quad (3.6)$$

Hence (3.3) reduces to

$$\Im \left[ -i\omega \lim_{u \rightarrow 1} \frac{1}{16\pi G_{d+1}} \int \frac{d^d q}{(2\pi)^d} \sqrt{\frac{g_{uu}}{g_{tt}}} \left( \sum_{i=1}^m N_i(u) \phi_i(u, \omega, q) \phi_i(u, -\omega, -q) \right) \right]. \quad (3.7)$$

Finally we get boundary action to be

$$S_\epsilon = \int \frac{d^d q}{(2\pi)^d} (\phi^i)^{(0)}(\omega, q) \mathcal{F}_{ik}(\omega, q) (\phi^k)^{(0)}(-\omega, -q), \quad (3.8)$$

where  $(\phi_i)^{(0)}$  is the boundary value of the field  $\phi_i$ . Next, the retarded correlators are given by

$$G^R = \begin{cases} -2\mathcal{F}_{ik}(\omega, q), & i = k, \\ -\mathcal{F}_{ik}(\omega, q), & i \neq k. \end{cases} \quad (3.9)$$

The expression for diagonal and off diagonal parts<sup>2</sup> of conductivity can be written as

$$\sigma_{ii} = - \lim_{\omega \rightarrow 0} \frac{\Im G^R(\omega, q=0)}{\omega} = \frac{1}{8\pi G_{d+1}} \left[ \frac{\sqrt{\frac{g_{uu}}{-g_{tt}}} \sum_{k=1}^m N_k(u) \phi_k(u, \omega, q) \phi_k(u, -\omega, -q)}{(\phi_i)^0 (\phi_i)^0} \right]_{u \rightarrow 1, q \rightarrow 0}, \quad (3.10)$$

$$\sigma_{ij} = - \lim_{\omega \rightarrow 0} \frac{\Im G^R(\omega, q=0)}{\omega} = \frac{1}{16\pi G_{d+1}} \left[ \frac{\sqrt{\frac{g_{uu}}{-g_{tt}}} \sum_{k=1}^m N_k(u) \phi_k(u, \omega, q) \phi_k(u, -\omega, -q)}{(\phi_i)^0 (\phi_j)^0} \right]_{u \rightarrow 1, q \rightarrow 0}. \quad (3.11)$$

respectively.

Now we are ready to compute the electrical conductivity. We first present computation for single charge black hole and show that we get the same result as obtained in [6] where conductivity was obtained by solving up to first order in  $\omega$  (note that here we only need to compute zeroth order in  $\omega$ ). Then we compute same for multiple charges in various dimensions.

---

<sup>2</sup>The coefficient of  $(\phi_i)^0 (\phi_j)^0$ .

## 4 Electrical conductivity

In this section we compute electrical conductivity for gauge theories dual to 4, 5, 7 dimensional R-charge black hole. We observe that behavior of conductivity with temperature is  $\sigma \sim T^{d-3}$  for d dimensional dual gauge theory. For multi-charge black hole we get conductivity matrix whose off diagonal parts comes solely due to effective interaction<sup>3</sup> between gauge fields. We first compute conductivity with single chemical potential and then turn to multiple chemical potential cases.

### 4.1 Single charge black hole in various dimension

For single charge black hole one finds

$$(\phi_1)'' + \left( \frac{f'}{f} + \frac{H_1'}{H_1} - \frac{c}{u} \right) (\phi_1)' - \frac{au^b(1+k_1)}{fH_1^2} (k_1\phi_1) = 0. \quad (4.1)$$

The expression for conductivity reduces to

$$\sigma = \frac{1}{8\pi G_{d+1}} \left[ \sqrt{\frac{g_{uu}}{-g_{tt}}} \frac{N(u)\phi(u, \omega, q)\phi(u, -\omega, -q)}{(\phi)^0(\phi)^0} \right]_{u \rightarrow 1, q \rightarrow 0}. \quad (4.2)$$

Let us define  $\mathcal{K} = \left[ \frac{1}{8\pi G_{d+1}} \sqrt{\frac{g_{uu}}{-g_{tt}}} N(u) \right]_{u \rightarrow 1}$  and  $F(u) = \phi_1(u, \omega, q)\phi_1(u, -\omega, -q)$ .

- **D=4:** In this case one gets  $c = 0, a = 1, b = 2$  and  $m = 0$ . Relevant parts are

$$\mathcal{K} = \frac{N_c^{\frac{3}{2}}}{24\sqrt{2}\pi} (1+k)^{\frac{3}{2}}. \quad (4.3)$$

$$\phi(u) = \phi^0 \frac{1 + \frac{2ku}{3}}{1 + ku} \quad (4.4)$$

which implies

$$\sigma = \mathcal{K} \frac{F(u=1)}{F(u=0)} = \frac{(3+2k)^2 N_c^{\frac{3}{2}}}{6^3 \pi \sqrt{2(1+k)}}. \quad (4.5)$$

We see that for three dimensional gauge theory, conductivity is independent of temperature. Now we can compare this result with the the result for  $\mu = 0$ , case.

$$\frac{\sigma_\mu}{\sigma_{\mu=0}} = \frac{(1 + \frac{2k}{3})^2}{\sqrt{1+k}} \geq 1. \quad (4.6)$$

---

<sup>3</sup>This can be interpreted as, turning on electric field for one type not only effects fields which are charged under that field, it also effects fields which are charged under other gauge fields as well, which is possible only when there exist interaction.

Since there exist a critical line  $k = \frac{3}{2}$  [6], one can not make conductivity arbitrarily large. This discussion also holds true for rest of the cases with only difference in location of critical line.

- **D=5:** Here  $c = 0, a = 1, b = 1$  and  $m = 0$ . Summary of the results are

$$\mathcal{K} = \frac{N_c^2 T_0 (1+k)^{\frac{3}{2}}}{16\pi}, \quad (4.7)$$

$$\phi(u) = \phi_0 \frac{1 + \frac{ku}{2}}{1 + ku}. \quad (4.8)$$

So one gets conductivity

$$\sigma = \frac{(2+k)^2 N_c^2 T_0}{64\pi \sqrt{(1+k)}} = \frac{N_c^2 T_H (2+k)}{32\pi}. \quad (4.9)$$

where  $T_H = \frac{(2+k)T_0}{2\sqrt{1+k}}$  is the hawking temperature of the black hole.

- **D=7:** In this case  $c = -1, a = 4, b = 3$  and  $m = 1$ . Relevant parts are

$$\mathcal{K} = \frac{4N_c^3 T_0^3 (1+k)^{\frac{3}{2}}}{81}, \quad (4.10)$$

$$\phi(u) = \phi_0 \frac{1 + \frac{ku^2}{3}}{1 + ku^2}. \quad (4.11)$$

Conductivity in this case is given by

$$\sigma = \frac{4(3+k)^2 N_c^3 T_0^3}{3^6 \sqrt{(1+k)}} = \frac{4N_c^3 T_H^3 (1+k)}{27(3+k)}. \quad (4.12)$$

where  $T_H = \frac{(3+k)T_0}{3\sqrt{1+k}}$  is the hawking temperature of the black hole.

## 4.2 Two charge black hole in various dimension

Now we turn to cases where two chemical potentials are turned on. Differential equations are<sup>4</sup>

$$\begin{aligned} (\phi_1)'' + \left( \frac{f'}{f} + 2\frac{H_1'}{H_1} - \frac{\mathcal{H}'}{\mathcal{H}} - \frac{c}{u} \right) (\phi_1)' \\ - \frac{au^b(1+k_1)(1+k_2)}{fH_1^2} \left[ k_1\phi_1 + \sqrt{k_1k_2} \phi_2 \right] = 0, \end{aligned} \quad (4.13)$$

---

<sup>4</sup>These form of equations are different than the form written in [5], because as mentioned earlier that we have done a rescaling.

and

$$(\phi_2)'' + \left( \frac{f'}{f} + 2\frac{H_2'}{H_2} - \frac{\mathcal{H}'}{\mathcal{H}} - \frac{c}{u} \right) (\phi_2)' - \frac{au^b(1+k_1)(1+k_2)}{fH_2^2} [k_2\phi_2 + \sqrt{k_1k_2}\phi_1] = 0, \quad (4.14)$$

Note that  $\mathcal{K}_i = \left[ \frac{1}{8\pi G_{d+1}} \sqrt{\frac{g_{uu}}{-g_{tt}}} N_i(u) \right]_{u \rightarrow 1}$  where  $N_i = \frac{fH_i^2}{u^m \mathcal{H}}$ . Now we compute case by case.

- **D=4:** Here one has  $c = 0, a = 1, b = 2$  and  $m = 0$ . Solutions are

$$\phi_1 = \frac{(a_0 + \frac{2a_0k_1 - b_0\sqrt{k_1k_2}}{3}u)}{1 + k_1u}, \quad \phi_2 = \frac{(b_0 + \frac{2b_0k_2 - a_0\sqrt{k_1k_2}}{3}u)}{1 + k_2u}, \quad (4.15)$$

and

$$\mathcal{K}_i = \frac{N_c^{\frac{3}{2}}(1+k_i)^2}{24\pi\sqrt{2(1+k_1)(1+k_2)}}. \quad (4.16)$$

Using these we get following form of conductivity.

$$\begin{pmatrix} \frac{N_c^{\frac{3}{2}}(9+(12+k_2)k_1+4k_1^2)}{6^3\pi\sqrt{2(1+k_1)(1+k_2)}} & -\frac{2N_c^{\frac{3}{2}}\sqrt{k_1k_2}(3+k_1+k_2)}{6^3\pi\sqrt{2(1+k_1)(1+k_2)}} \\ -\frac{2N_c^{\frac{3}{2}}\sqrt{k_1k_2}(3+k_1+k_2)}{6^3\pi\sqrt{2(1+k_1)(1+k_2)}} & \frac{N_c^{\frac{3}{2}}(9+(12+k_1)k_2+4k_2^2)}{6^3\pi\sqrt{2(1+k_1)(1+k_2)}} \end{pmatrix}.$$

- **D=5:** Here we have  $c = 0, a = 1, b = 1$  and  $m = 0$ . In this case solutions are

$$\phi_1 = \frac{(a_0 + \frac{a_0k_1 - b_0\sqrt{k_1k_2}}{2}u)}{1 + k_1u}, \quad \phi_2 = \frac{(b_0 + \frac{b_0k_2 - a_0\sqrt{k_1k_2}}{2}u)}{1 + k_2u}. \quad (4.17)$$

Where as

$$\mathcal{K}_i = \frac{N_c^2 T_0 (1+k_i)^2}{16\pi\sqrt{(1+k_1)(1+k_2)}}. \quad (4.18)$$

So we get conductivity as

$$\begin{pmatrix} \frac{(4+k_1^2+k_1(4+k_2))N_c^2 T_0}{64\pi\sqrt{(1+k_1)(1+k_2)}} & -\frac{(4+k_1+k_2)N_c^2 T_0}{64\pi} \sqrt{\frac{k_1k_2}{(1+k_1)(1+k_2)}} \\ -\frac{(4+k_1+k_2)N_c^2 T_0}{64\pi} \sqrt{\frac{k_1k_2}{(1+k_1)(1+k_2)}} & \frac{(4+k_2^2+k_2(4+k_1))N_c^2 T_0}{64\pi\sqrt{(1+k_1)(1+k_2)}} \end{pmatrix}.$$

So  $\sigma$  increases linearly with  $T_H$ .

- **D=7:** In this case  $c = -1, a = 4, b = 3$  and  $m = 1$ . Solutions are

$$\phi_1 = \frac{(a_0 + \frac{a_0k_1 - 2b_0\sqrt{k_1k_2}}{3}u^2)}{1 + k_1u^2}, \quad \phi_2 = \frac{(b_0 + \frac{b_0k_2 - 2a_0\sqrt{k_1k_2}}{3}u^2)}{1 + k_2u^2} \quad (4.19)$$

Now

$$\mathcal{K}_i = \frac{4N_c^3 T_0^3 (1+k_i)^2}{81\sqrt{(1+k_1)(1+k_2)}}. \quad (4.20)$$

Using these one finds conductivity matrix as

$$\begin{pmatrix} \frac{4(9+k_1(k_1+4k_2+6))N_c^3 T_o^3}{3^6 \sqrt{(1+k_1)(1+k_2)}} & -\frac{(6+k_1+k_2)N_c^3 T_o^3}{3^6} \sqrt{\frac{k_1 k_2}{(1+k_1)(1+k_2)}} \\ -\frac{(6+k_1+k_2)N_c^3 T_o^3}{3^6} \sqrt{\frac{k_1 k_2}{(1+k_1)(1+k_2)}} & \frac{4(9+k_2(k_2+4k_1+6))N_c^3 T_o^3}{3^6 \sqrt{(1+k_1)(1+k_2)}} \end{pmatrix}.$$

- Notice that off diagonal components of the conductivity matrix are negative but they are important, and plays crucial role when we construct thermal conductivity to viscosity ratio.
- Observe that off diagonal components goes as  $\sigma_{ij} \sim \Omega_i \Omega_j T^{d-3}$ , where  $\Omega_i = \frac{\mu_i}{2\pi T}$ . So switching off one of the chemical potential will make it zero, where as diagonal parts of conductivity goes as  $\sigma_{ii} \sim T^{d-3} f_{ii}(\Omega_1, \Omega_2)$ , where  $f_{ii}(0,0) \neq 0$  ( $\mu = 0$ , implies total charge density is zero i.e. there exist equal number of positive as well as negative charge and applying external electric field will induce flow of both in opposite direction which will contribute to electrical current). Since charged particles moves in opposite direction, there will be collisions among them and it ensures finite conductivity. As one increases  $\mu$ , conductivity should increase as relative number of collisions between opposite charges are less compared to zero chemical potential case.

### 4.3 Three charge black hole in various dimension

Now we turn to three charge black hole cases. General form of differential equations are

$$(\phi_i)'' + \left( \frac{f'}{f} + 2\frac{H'_i}{H_i} - \frac{\mathcal{H}'}{\mathcal{H}} \right) (\phi_i)' - \frac{u^b \prod_{j=1}^3 (1+k_j) \sqrt{k_i}}{f H_i^2} \left[ \sum_{j=1}^3 \sqrt{k_j} \phi_j \right] = 0. \quad (4.21)$$

- **D=4:** For this case the one gets  $b = 2$ . Relevant results in this case are

$$\phi_i = \frac{\left[ 3\phi_i^0 + \sqrt{k_i} \left( 3\sqrt{k_i} \phi_i^0 - \sum_{j=1}^3 \sqrt{k_j} \phi_j^0 \right) u \right]}{3(1+k_i u)}, \quad \mathcal{K}_i = \frac{N_c^{\frac{3}{2}} (1+k_i)^2}{24\pi \sqrt{2 \prod_{j=1}^3 (1+k_j)}}, \quad (4.22)$$

where  $\phi_i^0$  is the boundary value of perturbed gauge field.

Let us introduce,  $\sigma_{ij} = \frac{N_c^{\frac{3}{2}}}{6^3 \pi \sqrt{2(1+k_1)(1+k_2)(1+k_3)}} \sigma_{ij}^0$ , where  $\sigma_{ij}^0$ , is given by

$$\begin{pmatrix} 9 + (12 + k_2 + k_3)k_1 + 4k_1^2 & -\sqrt{k_1 k_2}(6 + 2k_1 + 2k_2 - k_3) & -\sqrt{k_1 k_3}(6 + 2k_1 + 2k_3 - k_2) \\ -\sqrt{k_1 k_2}(6 + 2k_1 + 2k_2 - k_3) & 9 + (12 + k_1 + k_3)k_2 + 4k_2^2 & -\sqrt{k_2 k_3}(6 + 2k_2 + 2k_3 - k_1) \\ -\sqrt{k_1 k_3}(6 + 2k_1 + 2k_3 - k_2) & -\sqrt{k_2 k_3}(6 + 2k_2 + 2k_3 - k_1) & 9 + (12 + k_1 + k_2)k_3 + 4k_3^2 \end{pmatrix}.$$

- **D=5:** For this case  $b = 1$ . Results needed for conductivity calculation are

$$\phi_i = \frac{\left[ 2\phi_i^0 + \sqrt{k_i} \left( 2\sqrt{k_i} \phi_i^0 - \sum_{j=1}^3 \sqrt{k_j} \phi_j^0 \right) u \right]}{2(1 + k_i u)}, \quad \mathcal{K}_i = \frac{N_c^2 T_0 (1 + k_i)^2}{16\pi \sqrt{(1 + k_1)(1 + k_2)(1 + k_3)}}. \quad (4.23)$$

Defining as before  $\sigma_{ij} = \frac{N_c^2 T_0}{64\pi \sqrt{(1+k_1)(1+k_2)(1+k_3)}} \sigma_{ij}^0$ , where  $\sigma_{ij}^0$  is given by

$$\begin{pmatrix} 4 + k_1^2 + k_1(4 + k_2 + k_3) & -\sqrt{k_1 k_2}(4 + k_1 + k_2 - k_3) & -\sqrt{k_1 k_3}(4 + k_1 + k_3 - k_2) \\ -\sqrt{k_1 k_2}(4 + k_1 + k_2 - k_3) & 4 + k_2^2 + k_2(4 + k_1 + k_3) & -\sqrt{k_2 k_3}(4 + k_2 + k_3 - k_1) \\ -\sqrt{k_1 k_3}(4 + k_1 + k_3 - k_2) & -\sqrt{k_2 k_3}(4 + k_2 + k_3 - k_1) & 4 + k_3^2 + k_3(4 + k_1 + k_2) \end{pmatrix}$$

## 4.4 Four charge black hole

Differential equations are

$$(\phi_i)'' + \left( \frac{f'}{f} + 2\frac{H'_i}{H_i} - \frac{\mathcal{H}'}{\mathcal{H}} \right) (\phi_i)' - \frac{u^2 \prod_{j=1}^4 (1 + k_j) \sqrt{k_i}}{f H_i^2} \left[ \sum_{j=1}^4 \sqrt{k_j} \phi_j \right] = 0, \quad (4.24)$$

and solutions are

$$\phi_i = \frac{\left[ 3\phi_i^0 + \sqrt{k_i} \left( 3\sqrt{k_i} \phi_i^0 - \sum_{j=1}^4 \sqrt{k_j} \phi_j^0 \right) u \right]}{3(1 + k_i u)}, \quad \mathcal{K}_i = \frac{N_c^{\frac{3}{2}} (1 + k_i)^2}{24\pi \sqrt{2 \prod_{j=1}^4 (1 + k_j)}}. \quad (4.25)$$

Using these we get following form of conductivity.

$$\sigma_{ij} = \frac{N_c^{\frac{3}{2}}}{6^3 \pi \sqrt{2(1+k_1)(1+k_2)(1+k_3)(1+k_4)}} \sigma_{ij}^0. \quad (4.26)$$

Where

$$\sigma_{ii}^0 = 9 + \left(12 + \sum_{j=1}^4 k_j\right) k_i + 3k_i^2 \quad \text{and} \quad \sigma_{ij}^0 = -\left(6 - \sum_{l=1}^4 k_l + 3(k_i + k_j)\right).$$

- **Some special cases :** Using above results one can study special cases such as effect of small chemical potential or the case with equal chemical potential. Note that in the case when all the chemical potential are equal then there exist no second order phase transition. In this case, temperature i.e.  $T \geq 0$  gives a constraint on the possible maximum value of chemical potential.

Dimension	Constraint( $T \geq 0$ )	$\sigma$
5	$k \leq 2$	$\frac{3(2-k)^2 N_c^2 T_0}{32\pi(1+k)^{\frac{3}{2}}}$
4	$k \leq 3$	$\frac{(3-k)^2 N_c^{\frac{3}{2}}}{27\sqrt{2}(1+k)^2 \pi}$
7	$k \leq 3$	$\frac{16(3-k)^2 N_c^3 T_0^3}{729(1+k)}$

Note that as  $T \rightarrow 0$ ,  $\sigma \rightarrow 0$  quadratically in the parameter  $k$ , irrespective of which dimension we are in<sup>5</sup>.

## 5 General analysis for electrical conductivity

In the absence of chemical potential the general form of conductivity was written down in [10] using membrane paradigm at the horizon (see also [11], [12] for other general discussions). But in the presence of chemical potential one can't use those formulas as there is a nontrivial flow from horizon to boundary. To obtain a general form of conductivity, we analyze the most general form of the perturbation equation both near the horizon as well as near the boundary and use these information's to fix the form of conductivity. Let us focus at the single charge case first.

---

<sup>5</sup>Determinant of conductivity matrix for general  $\mu$  also vanishes in similar way once we approach extremality even for M2 brane case where conductivity is independent of temperature.

- **Single charge :** In this case equation is

$$\frac{d^2}{du^2}\phi^i(u) + \left(\frac{d}{du}\log(\sqrt{-g}G_{ii}g^{xx}g^{uu})\right)\frac{d}{du}\phi^i(u) + \frac{1}{g_{tt}}G_{ii}(F_{ut}^i)^2\phi^i(u) = 0, \quad (5.1)$$

Near the horizon ( $u \rightarrow 1$ ),  $N_i = \sqrt{-g}G_{ii}g^{xx}g^{uu} \rightarrow (1-u)A$ , and  $\frac{1}{g_{tt}}G_{ii}(F_{ut}^i)^2 \rightarrow -\frac{F}{1-u}$  where  $A, F$  are some constant<sup>6</sup>. To the leading order in inverse power of  $(1-u)$ , one gets

$$\frac{d}{du}\phi^i + F\phi^i = 0. \quad (5.2)$$

Using this the general form of solution can be given by

$$\phi^i(u) = a_0 \frac{nH_i + (H_iF - H'_i)(1-u^n)}{H_i(u)(H_iF + nH_i - H'_i)} \quad (5.3)$$

Where  $H_i, H'_i$  are evaluated at the horizon<sup>7</sup>( $u = 1$ ). The general form of  $\sigma$  for single charge case can be written as

$$\sigma = \frac{1}{8\pi G_{d+1}}G_{II}g^{xx}\sqrt{\frac{-g}{g_{uu}g_{tt}}}\left(\frac{n}{H_iF + nH_i - H'_i}\right)^2, \quad (5.5)$$

where every quantity in the above is calculated at the horizon<sup>8</sup>( $u = 1$ ). Value of  $n$  can be determined by looking at

$$\frac{d}{du}(N_i \frac{d}{du}\phi^i(u)) + \sum_{j=1}^m M_{ij}\phi^j(u) = 0, \quad (5.6)$$

near the boundary i.e.  $u \sim 0$ , if

$$N_i \sim u^{1-n} \quad (5.7)$$

then

$$\phi^i(u) \sim a_0 + a_1 u^n. \quad (5.8)$$

As an example consider single charge  $M5 - brane$  case. Here

$$N_1 \sim u^{-1} \Rightarrow n = 2 \quad \text{and} \quad F = \frac{4k_1(1+k_1)}{(3+k_1)H_1^2}$$

which after plugging on the expression for conductivity one finds same result as in section 4.1. Note that explicit solution in the case of  $M5 - brane$  has  $\phi(u) = \phi_0 \frac{1+\frac{ku^2}{3}}{1+ku^2}$  i.e.  $n = 2$ , which is consistent with our analysis.

---

<sup>6</sup>Lets take  $g_{tt} = (1-u)\gamma_0$ , then  $F = (\frac{1}{\gamma_0}G_{ii}(F_{ut}^i)^2)|_{u=1}$ .

<sup>7</sup>Note that  $H_i(u)$  can be replaced by

$$\frac{\sqrt{\sqrt{g}G_{ii}g^{uu}g^{tt}u^{n-1}}}{\sqrt{\sqrt{g}G_{ii}g^{uu}g^{tt}u^{n-1}}|_{u=0}}. \quad (5.4)$$

, which makes expression for conductivity independent of background.

<sup>8</sup>In all the cases considered above  $nH_i - H'_i = n$ . In the absence of chemical potential  $F = 0$  so one obtains the usual result as written in [10], see also [9] for related discussion in the presence of chemical potential.

- **Multicharge case:** Now we analyze multi charge black hole (one can do an analogous analysis as done above for multicharge case, but here we avoid repetition). The conductivity matrix that we obtained is symmetric, hence one can go to a suitable basis where conductivity is diagonal. In this basis there is no effective interaction between different fields, in other words they behave as independent field. Which implies we can apply same general formula for conductivity as for single charge case. As an example lets take  $D3$  brane 2 charge case. Here the diagonal form of the conductivity is given by

$$\begin{pmatrix} \frac{N_c^2 T_o}{16\pi\sqrt{(1+k_1)(1+k_2)}} & 0 \\ 0 & \frac{(2+k_1+k_2)^2 N_c^2 T_o}{64\pi\sqrt{(1+k_1)(1+k_2)}} \end{pmatrix}$$

Expression for charge density and chemical potential in this basis are given by

$$\begin{aligned} \varrho_1 &= \frac{\sqrt{k_2}\rho_1}{\sqrt{k_1+k_2}} + \frac{\sqrt{k_1}\rho_2}{\sqrt{k_1+k_2}}, & \varrho_2 &= -\frac{\sqrt{k_1}\rho_1}{\sqrt{k_1+k_2}} + \frac{\sqrt{k_2}\rho_2}{\sqrt{k_1+k_2}} \\ \tilde{\mu}_1 &= \frac{\sqrt{k_2}\mu_1}{\sqrt{k_1+k_2}} + \frac{\sqrt{k_1}\mu_2}{\sqrt{k_1+k_2}}, & \tilde{\mu}_2 &= -\frac{\sqrt{k_1}\mu_1}{\sqrt{k_1+k_2}} + \frac{\sqrt{k_2}\mu_2}{\sqrt{k_1+k_2}}. \end{aligned} \quad (5.9)$$

One can start with differential equation and do a same rotation as above to go to a diagonal basis and using general results of single charge, can obtain same values of conductivities as appeared in the diagonal elements.

## 6 Thermal conductivity

In this section we first review hydrodynamics with multiple conserved charges and write down expression for thermal conductivity, using which we compute thermal conductivity and show that thermal conductivity and viscosity obeys Wiedemann-Franz like law even in the presence of multiple chemical potential. Single charge case was discussed in [1] .

- **Relativistic hydrodynamics with multiple conserved charge:** The continuity equations are

$$\partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu J_i^\mu = 0 \quad (6.1)$$

where

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \tau^{\mu\nu}, \quad J_i^\mu = \rho_i u^\mu + \nu_i^\mu \quad (6.2)$$

In the above  $\epsilon$  and  $P$  are the local energy density and pressure,  $u^\mu$  is the local velocity and it obeys  $u_\mu u^\mu = -1$ , where as  $\tau^{\mu\nu}$  and  $\nu_i^\mu$  are the dissipative parts of stress-energy tensor and current.

Following [1] one can choose  $u^\mu$  and  $\rho_i$ 's so that

$$u_\mu \tau^{\mu\nu} = u_\mu \nu_i^\mu = 0. \quad (6.3)$$

Note that

$$u_\nu \partial_\mu T^{\mu\nu} = -(\epsilon + P) \partial_\mu u^\mu - u^\mu \partial_\mu \epsilon + u_\nu \partial_\mu \tau^{\mu\nu} = 0 \quad (6.4)$$

We also have

$$\epsilon + P = Ts + \sum_{i=1}^m \mu^i \rho_i, \quad d\epsilon = Tds + \sum_{i=1}^m \mu^i d\rho_i \quad (6.5)$$

Using which one gets

$$\partial_\mu \left( s u^\mu - \sum_{i=1}^m \frac{\mu^i}{T} \nu_i^\mu \right) = - \sum_{i=1}^m \nu_i^\mu \partial_\mu \frac{\mu^i}{T} - \frac{\tau^{\mu\nu}}{T} \partial_\mu u^\nu. \quad (6.6)$$

Now  $\partial_\mu \left( s u^\mu - \sum_{i=1}^m \frac{\mu^i}{T} \nu_i^\mu \right)$  can be interpreted as the divergence of the entropy current which implies right hand side is positive. So we write

$$\nu_i^\mu = - \sum_{j=1}^m \varkappa_{ij} \left( \partial^\mu \frac{\mu^j}{T} + u^\mu u^\lambda \partial_\lambda \frac{\mu^j}{T} \right) \quad (6.7)$$

and similarly for  $\tau^{\mu\nu}$  (see [1]). To interpret  $\varkappa_{ij}$  as the coefficient of thermal conductivity, consider no charge current i.e.  $J_j^\alpha = 0$ ,<sup>9</sup> but there is an energy flow,  $T^{t\alpha} \neq 0$ , which is the heat flow. Take  $u^\alpha$  to be small so that one gets using eqn. (6.2)

$$\rho_i u^\alpha = \sum_{j=1}^m \varkappa_{ij} \partial^\alpha \frac{\mu^j}{T}.$$

From which one can obtain

$$\sum_{i,j=1}^m \rho_i \varkappa_{ij}^{-1} \rho_j u^\alpha = \sum_{i=1}^m \rho_i \partial^\alpha \frac{\mu^i}{T}, \quad (6.8)$$

hence

$$u^\alpha = \frac{1}{\sum_{i,j=1}^m \rho_i \varkappa_{ij}^{-1} \rho_j} \sum_{l=1}^m \rho_l \partial^\alpha \frac{\mu^l}{T}. \quad (6.9)$$

Using eqn. (6.5) we get

$$\sum_{i=1}^m \rho_i \partial^\alpha \frac{\mu^i}{T} = -\frac{\epsilon + P}{T^2} \partial^\alpha T + \frac{\partial^\alpha P}{T} \quad (6.10)$$

---

<sup>9</sup>In our notation  $\mu, \nu$  runs from  $t, 1, 2, \dots, D$ , where as  $\alpha$  runs from  $1, 2, \dots, D$ , and  $i, j$  are R-charge indices.

after substitution, this gives

$$u^\alpha = -\frac{1}{\sum_{i,j=1}^m \rho_i \chi_{ij}^{-1} \rho_j} \left( \frac{\epsilon + P}{T^2} \right) (\partial^\alpha T - \frac{T}{\epsilon + P} \partial^\alpha P). \quad (6.11)$$

Therefore

$$T^{t\alpha} = (\epsilon + P)u^\alpha = -\frac{1}{\sum_{i,j=1}^m \rho_i \chi_{ij}^{-1} \rho_j} \left( \frac{\epsilon + P}{T} \right)^2 (\partial^\alpha T - \frac{T}{\epsilon + P} \partial^\alpha P) \quad (6.12)$$

hence the coefficient of thermal conductivity can be read off as

$$\kappa_T = \left( \frac{\epsilon + P}{T} \right)^2 \frac{1}{\sum_{i,j=1}^m \rho_i \chi_{ij}^{-1} \rho_j}. \quad (6.13)$$

Note that,  $\chi_{ij}$  can be found out from greens function as

$$G_{ij}(\omega, q = 0) = -i\omega \frac{\chi_{ij}}{T}, \quad (6.14)$$

where  $J_i^\alpha = -G_{ij}(\omega, q = 0)A_j^\alpha$  and  $G_{ij}(\omega, q = 0)$  can be obtained using kubo formula as done in previous sections. Now we turn to computation of  $\kappa_T$ , and dimension less ratio <sup>10</sup>

$$\frac{\kappa_T}{\eta T} \sum_{j=1}^m (\mu^j)^2 \quad (6.15)$$

case by case.

- **Single charge black hole:** Note that for single charge black hole  $\frac{1}{\rho_i \chi_{ij}^{-1} \rho_j} = \frac{\chi}{\rho^2}$ . So that one gets

$$\kappa_T = \left( \frac{\epsilon + P}{\rho T} \right)^2 \chi = \left( \frac{\epsilon + P}{\rho} \right)^2 \frac{\sigma}{T} \quad (6.16)$$

Rather than providing details we here tabulate the thermal conductivity and dimension less ratio  $\frac{\kappa_T \mu^2}{\eta T}$ , where  $\eta$  is the shear viscosity.

---

<sup>10</sup>There exist interesting relationship

$$\frac{\sum_{j=1}^m (\mu^j)^2}{\sum_{i,j=1}^m \rho_i \chi_{ij}^{-1} \rho_j} = \frac{\sum_{i,j=1}^m \mu_i \chi_{ij} \mu_j}{\sum_{j=1}^m (\rho_j)^2}$$

in all the example considered in the text

Dimension	$\kappa_T$	$\frac{\kappa_T \mu^2}{\eta T}$
5	$\frac{(1+k)^2 N_c^2 T^2 \pi}{k(2+k)}$	$8\pi^2$
4	$\frac{2\sqrt{2}(1+k)^{3/2} N_c^{3/2} T \pi}{3k}$	$32\pi^2$
7	$\frac{8(1+k)^3 N_c^3 T^4 \pi^2}{3k(3+k)^3}$	$2\pi^2$

- **Two charge black hole:** In the ratio  $\frac{\kappa_T \mu^2}{\eta T}$ ,  $\mu^2$  is replaced by  $\mu_1^2 + \mu_2^2$ . Note that  $\mu_i \rightarrow -\mu_i$  is a symmetry <sup>11</sup> which implies reversing the sign of charge density.

Dimension	$\kappa_T$	$\frac{\kappa_T(\mu_1^2 + \mu_2^2)}{\eta T}$
5	$\frac{N_c^2 T^2 \pi}{\left(2 + \sum_{j=1}^2 k_j\right) \left(\sum_{j=1}^2 \frac{k_j}{(1+k_j)^2}\right)}$	$8\pi^2$
4	$\frac{(2N_c)^{\frac{3}{2}} T \pi}{\left(\sum_{j=1}^2 \frac{3k_j}{(1+k_j)^2}\right) \sqrt{\prod_{i=1}^2 (1+k_i)}}$	$32\pi^2$
7	$\frac{(2N_c)^3 T^4 \pi^2 \prod_{i=1}^2 (1+k_i)}{\left(\sum_{j=1}^2 \frac{3k_j}{(1+k_j)^2}\right) \left(3 + \sum_{j=1}^2 k_j - \prod_{j=1}^2 k_j\right)^3}$	$2\pi^2$

- **Three charge black hole:**

Dimension	$\kappa_T$	$\frac{\kappa_T(\mu_1^2 + \mu_2^2 + \mu_3^2)}{\eta T}$
5	$\frac{N_c^2 T^2 \pi}{\left(2 + \sum_{j=1}^3 k_j - \prod_{j=1}^3 k_j\right) \left(\sum_{j=1}^3 \frac{k_j}{(1+k_j)^2}\right)}$	$8\pi^2$
4	$\frac{(2N_c)^{\frac{3}{2}} T \pi}{\left(\sum_{j=1}^3 \frac{3k_j}{(1+k_j)^2}\right) \sqrt{\prod_{i=1}^3 (1+k_i)}}$	$32\pi^2$

- **Four charge black hole (4 Dimensional black hole):** The thermal conductivity is given by

$$\kappa_T = \frac{(2N_c)^{\frac{3}{2}} T \pi}{\left(\sum_{j=1}^4 \frac{3k_j}{(1+k_j)^2}\right) \sqrt{\prod_{i=1}^4 (1+k_i)}} \quad (6.17)$$

---

<sup>11</sup>The expression for  $\kappa_T$  in eqn. (6.13) is invariant under  $SO(m)$  rotation among  $\rho_i$ 's.

and

$$\frac{\kappa_T(\mu_1^2 + \mu_2^2 + \mu_3^2 + \mu_4^2)}{\eta T} = 32\pi^2. \quad (6.18)$$

- $\mu = 0$  : We observe that irrespective of number of chemical potential turned on, thermal conductivity to viscosity ratio shows same value. We also observe that as  $\mu \rightarrow 0$  i.e.  $\rho \rightarrow 0$ , thermal conductivity given in eqn. (6.16), diverges which implies finite, non decaying momentum (see [13], for more details).

## 7 Conductivity for extremal black hole

In this section we compute electrical conductivity for extremal background. At extremality metric in the vicinity of horizon takes the form

$$\bar{g}_{tt} = -f(u)A_1(u) = -(1-u)^2\gamma_0, \quad \bar{g}_{uu} = A_2(u)f^{-1}(u) = \frac{\gamma_u}{(1-u)^2}, \quad f(u) = (1-u)^2V(u). \quad (7.1)$$

where  $\gamma_0$  and  $\gamma_u$  are some constants. Near the horizon eqn. (2.10) reduces to

$$\frac{d^2}{du^2}\phi_i(u) - \frac{2}{1-u}\frac{d}{du}\phi_i(u) + \frac{\gamma_u}{\gamma_0}\frac{\omega^2}{(1-u)^4}\phi_i(u) - \frac{c_i}{(1-u)^2}\frac{(\sum_{j=1}^m d_j\phi_j(u))}{\gamma_0} = 0 \quad (7.2)$$

Note that  $c_i = F_{ut}^i(u=1)$  and  $d_j = G_{jj}(u)F_{ut}^j(u)$  at  $u=1$ . Following [7], [8] let us define  $u = 1 - \frac{\omega}{\xi}$ . In this coordinate system eqn. (7.2) reduces to

$$\frac{d^2}{d\xi^2}\phi_i(\xi) + \frac{\gamma_u}{\gamma_0}\phi_i(\xi) - \frac{c_i}{\xi^2}\frac{(\sum_{j=1}^m d_j\phi_j(\xi))}{\gamma_0} = 0 \quad (7.3)$$

Above equation is in general a complicated coupled differential equation. To solve it we observe that

$$\frac{\frac{d^2}{d\xi^2}\phi_i(\xi) + \frac{\gamma_u}{\gamma_0}\phi_i(\xi)}{c_i} = \frac{(\sum_{j=1}^m d_j\phi_j(\xi))}{\gamma_0\xi^2} \quad (7.4)$$

In the case when more than one field is present then we get

$$\frac{\frac{d^2}{d\xi^2}\phi_1(\xi) + \frac{\gamma_u}{\gamma_0}\phi_1(\xi)}{c_1} = \frac{\frac{d^2}{d\xi^2}\phi_2(\xi) + \frac{\gamma_u}{\gamma_0}\phi_2(\xi)}{c_2} = \dots \quad (7.5)$$

We take solution of the form

$$\frac{\phi_1(\xi)}{c_1} = \frac{\phi_2(\xi)}{c_2} = \dots \quad (7.6)$$

Plugging eqn. (7.6) in eqn. (7.3) one obtains

$$\frac{d^2}{d\xi^2}\phi_i(\xi) + \frac{\gamma_u}{\gamma_0}\phi_i(\xi) - \frac{(\sum_{j=1}^m d_j c_j)}{\gamma_0 \xi^2}\phi_i(\xi) = 0 \quad (7.7)$$

Introduce  $\eta = \sqrt{\frac{\gamma_u}{\gamma_0}} \xi$  and  $a = \frac{(\sum_{j=1}^m d_j c_j)}{\gamma_0}$ , so that one gets ( from eqn.(7.7))

$$\frac{d^2}{d\eta^2}\phi_i(\eta) + \phi_i(\eta) - \frac{a}{\eta^2}\phi_i(\eta) = 0 \quad (7.8)$$

The incoming solution to eqn. (7.8) takes the form

$$\phi_i(\eta) = CH_\nu^1(\eta), \quad (7.9)$$

where  $H_\nu^1(\eta)$  is henkel function and  $\nu = \frac{\sqrt{1+4a}}{2}$ . Taking  $\eta \rightarrow 0$  limit one gets

$$\lim_{\eta \rightarrow 0} \phi_i(\eta) = \eta^{\frac{1}{2}+\nu} 2^{-\nu} \left( \frac{1}{\Gamma[1+\nu]} - i \frac{\cos(\pi\nu)\Gamma[-\nu]}{\pi} \right) - i\eta^{\frac{1}{2}-\nu} 2^\nu \frac{\Gamma[\nu]}{\pi} \quad (7.10)$$

Using  $\eta = \sqrt{\frac{\gamma_u}{\gamma_0}} \frac{\omega}{(1-u)}$ , and some properties of Gamma functions as well as doing some rescaling one finds

$$\phi_i(u \rightarrow 1) = A_0 \left[ \frac{1}{(1-u)^{\frac{1}{2}-\nu}} + \left( \sqrt{\frac{\gamma_u}{\gamma_0}} \right)^{2\nu} \left( \frac{\omega}{2} \right)^{2\nu} \frac{\pi(i - \cot(\nu\pi))}{\Gamma[1+\nu]\Gamma[\nu]} \frac{1}{(1-u)^{\frac{1}{2}+\nu}} \right]. \quad (7.11)$$

Again using properties of Gamma functions we get

$$\phi_i(u \rightarrow 1) = A_0 \left[ \frac{1}{(1-u)^{\frac{1}{2}-\nu}} - \left( \sqrt{\frac{\gamma_u}{\gamma_0}} \right)^{2\nu} \left( \frac{\omega}{2} \right)^{2\nu} \frac{\Gamma[1-\nu]}{\Gamma[1+\nu]} \frac{e^{-i\nu\pi}}{(1-u)^{\frac{1}{2}+\nu}} \right]. \quad (7.12)$$

Upon comparing with standard form [8] we conclude

$$g(\omega) = -2\nu e^{-i\nu\pi} \left( \sqrt{\frac{\gamma_u}{\gamma_0}} \right)^{2\nu} \left( \frac{\omega}{2} \right)^{2\nu} \frac{\Gamma[1-\nu]}{\Gamma[1+\nu]} \quad (7.13)$$

which corresponds to greens function of operator of dimension  $\delta = \nu + \frac{1}{2}$ . We obtain conductivity to be proportional to

$$\sigma \propto \lim_{\omega \rightarrow 0} \frac{1}{\omega} \Im[g(\omega)] \propto (\omega)^{2\nu-1}, \quad (7.14)$$

where

$$\begin{aligned} 2\nu &= \sqrt{1+4a} \\ &= \sqrt{1 + \left( \frac{4}{\gamma_0} \right) \sum_{j=1}^m d_j c_j} \\ &= \sqrt{1 + \left( \frac{4}{\gamma_0} \right) \sum_{j=1}^m G_{jj} (F_{ut}^j)^2}. \end{aligned} \quad (7.15)$$

In the above expression every quantity is calculated at the horizon ( $u = 1$ ). Hence, we see only way to get non-zero conductivity in the limit  $\omega \rightarrow 0$  at extremality is  $\nu \leq \frac{1}{2}$  where as  $\sigma \rightarrow 0$  if  $\nu > \frac{1}{2}$ .

- To obtain above form of  $g(\omega)$ , we have only assumed that extremal black hole exhibits double pole. So the expression for operator dimension in general follows only from criteria of extremality i.e. it is independent of particular background. In all the examples considered below we find  $\nu = \frac{3}{2} \Rightarrow \delta = \nu + \frac{1}{2} = 2$ . There are other classes of black hole as well (dilatonic black hole [14]) where one finds  $\delta = 2$ .

- **Four dimensional black hole:** In this case

$$2\nu = \sqrt{1 + 4 \frac{\prod_{i=1}^4 (1 + k_i)}{3 + \sum_{j=1}^4 k_j + \prod_{i=1}^4 k_i} \left( \sum_{j=1}^4 \frac{k_i}{(1 + k_i)^2} \right)} \quad (7.16)$$

Using extremality condition<sup>12</sup>(see appendix) we get  $2\nu = 3$ .

- **Five dimensional black hole:** In this case

$$2\nu = \sqrt{1 + 4 \frac{\prod_{i=1}^3 (1 + k_i)}{1 + \prod_{i=1}^3 k_i} \left( \sum_{j=1}^3 \frac{k_i}{(1 + k_i)^2} \right)} \quad (7.17)$$

Using extremality condition<sup>13</sup> one finds  $2\nu = 3$ . Which implies  $\delta = \nu + \frac{1}{2} = 2$ . Note that above result also applicable for 5d Rn black hole (for which  $k_1 = k_2 = k_3$ ) considered in other places [7].

- **Seven dimensional black hole:** In this case

$$2\nu = \sqrt{1 + 4 \frac{4(1 + k_1)(1 + k_2)}{3 + k_1 k_2} \left( \frac{k_1}{(1 + k_1)^2} + \frac{k_2}{(1 + k_2)^2} \right)} \quad (7.18)$$

Now extremality condition implies  $k_1 = \frac{3+k_2}{k_2-1}$ . So one gets  $2\nu = 3$ .

---

<sup>12</sup>  $k_1 = \frac{3+2(k_2+k_3+k_4)+k_2(k_3+k_4)+k_3k_4}{k_2k_3k_4-2-k_2-k_3-k_4}$

<sup>13</sup>  $k_3 = \frac{2+k_1+k_2}{k_1k_2-1}$

- Above results implies that for black hole at extremality obeys

$$\left(\frac{1}{\gamma_0}\right) \sum_{j=1}^m G_{jj} (F_{ut}^j)^2 = 2. \quad (7.19)$$

It would be interesting to find out under what conditions extremal backgrounds obeys this relation.

## 8 Schrödinger like equation with effective potential

One can gain much clearer understanding by reducing the problem of finding conductivity to solving equivalent Schrödinger problem [14]. General form of perturbation equation in  $d + 1$  dimensional gravity theory (d-dimensional gauge theory) with multiple chemical potential are

$$\frac{d}{du} \left( N_i \frac{d}{du} \phi_i(u) \right) - \omega^2 N_i g_{uu} g^{tt} \phi_i(u) + \sum_{j=1}^m M_{ij} \phi_j(u) = 0. \quad (8.1)$$

with

$$N_i = \sqrt{-g} G_{ii} g^{xx} g^{uu}. \quad (8.2)$$

and

$$M_{ij} = F_{ut}^i \sqrt{-g} G_{ii} g^{xx} g^{uu} g^{tt} G_{jj} F_{ut}^j. \quad (8.3)$$

Define  $f_i^2 = G_{ii}$ ,  $h_i^2 = f_i^2 g_{xx}^{\frac{d-3}{2}}$ , new variable  $\frac{d}{dz} = \sqrt{-\frac{g_{tt}}{g_{uu}}} \frac{d}{du}$ , and new wave function  $\Psi_i = h_i \phi_i$ . Using above definitions we get

$$-\Psi_i'' + \sum_{j=1}^m V_{ij} \Psi_j = \omega^2 \Psi_i, \quad (8.4)$$

with

$$V_{ij} = \frac{h_i''}{h_i} \delta_{ij} - f_i f_j g^{tt} A_t'^i A_t'^j, \quad (8.5)$$

where ' denotes derivative with respect to  $z$  coordinates and  $A_t^i$  's are background gauge fields. Since  $V_{ij}$  is symmetric one can do  $So(N)$  rotation among these fields  $\Psi$  to go to a diagonal basis where one gets decoupled Schrödinger equation. In the remaining part of this section, after commenting about the form of the potential at the horizon and conductivity in terms of reflection coefficient we turn to conductivity for extremal black holes.

- **Potential at the horizon:** Regularity at the horizon implies

$$\lim_{u \rightarrow 1} \frac{d}{du} \phi_i(u) = -i\omega \lim_{u \rightarrow 1} \sqrt{-\frac{g_{uu}}{g_{tt}}} \phi_i(u) + \mathcal{O}(\omega^2). \quad (8.6)$$

Now using,  $\frac{d}{dz} = \sqrt{-\frac{g_{tt}}{g_{uu}}} \frac{d}{du}$  at the horizon we get

$$\frac{d}{dz}\Psi_i(z) = -i\omega\Psi_i(z) + \mathcal{O}(\omega^2). \quad (8.7)$$

Plugging this in equation (8.4) we conclude  $V = 0$ , at the horizon.

Now we can follow standard procedure as in [14] to interpret conductivity in terms of reflection coefficient.<sup>14</sup> As we approach horizon we expect that reflection coefficient should decrease and hence  $\sigma_{Horizon} \geq \sigma_{Boundary}$  (as shown in the text). It will be interesting to explore further the Schrödinger potential in detail, and extract out conductivity away from zero frequency limit.

- **Extremal case:** In the inner region (close to horizon) for extremal black hole one finds (see sec.7)

$$\frac{d^2}{d\eta^2}\phi_i(\eta) + \phi_i(\eta) - \frac{a}{\eta^2}\phi_i(\eta) = 0.$$

Since  $\eta$  contains a factor of  $\omega$ , we define  $x = \omega\eta$ . In this coordinate we get

$$\frac{d^2}{dx^2}\phi_i(x) + \omega^2\phi_i(x) - \frac{a}{x^2}\phi_i(x) = 0.$$

Hence the potential is given by

$$V_{Nearhorizon}(x) = \frac{a}{x^2} \quad (8.8)$$

At the horizon

$$\lim_{x \rightarrow \infty} V(x) = 0, \quad (8.9)$$

which is in agreement with non-extremal case (where  $x \rightarrow \infty$ , corresponds to horizon). Following argument in [14] we conclude at extremality<sup>15</sup>  $Re(\sigma) = \omega^a$ . Note that  $a = 2$ , for all the examples considered in the text.

## 9 Discussion

Some of the issues that are studied and worth exploring are

---

<sup>14</sup>If we follow effective action approach as developed in section 3, we get  $\sigma(\omega \rightarrow 0) = \frac{|\Psi(z \rightarrow horizon)|^2}{|\Psi(z \rightarrow boundary)|^2} - \frac{1}{i\omega} \frac{h'}{h}$ . Now using [14]  $\Psi(z) = e^{-i\omega z} + Re^{i\omega z}$  for  $z > 0$  and  $\Psi(z) = Te^{-i\omega z}$  near the horizon, we get  $\sigma = \frac{T^2}{(1+R)^2}$ . Using  $T^2 + R^2 = 1$ , we reach at  $\sigma = \frac{1-R}{1+R}$ .

<sup>15</sup>Author is thankful to referee for pointing this out.

- Using our general results on perturbation equations we found out an expression for electrical conductivity for general background and they are different than  $\mu = 0$  cases. Effect of turning on chemical potential can be thought of as turning on effective interaction in the back ground. Note that conductivity satisfies  $\sigma_{Horizon} > \sigma_{Boundary}$ . More precisely variation of  $\sigma$  with  $u$  is given by

$$\sigma(u) = \frac{1}{8\pi G_{d+1}} G_{II} g^{xx} \sqrt{\frac{-g}{g_{uu}g_{tt}}} \left[ \frac{nH_i(u)}{nH_i + (H_i F - H'_i)(1 - u^n)} \right]^2. \quad (9.1)$$

At the horizon one gets

$$\sigma = \frac{1}{8\pi G_{d+1}} G_{II} g^{xx} \sqrt{\frac{-g}{g_{uu}g_{tt}}}. \quad (9.2)$$

One can arrive at the form of the conductivity at the horizon using membrane paradigm arguments as developed in [10].

- In the extremal case it is shown that at the horizon potential remains finite and following [14] it may be concluded that conductivity is zero in this case. We also observe that dimension of dual operator can be given purely in terms of geometry and gauge fields. To arrive at that form we only have assumed double pole structure of the metric. So we expect result is independent of back ground. In all the examples considered, the dimension of dual operator is always 2.
- It is also shown that thermal conductivity and viscosity obeys Wiedemann-Franz like law even in the presence of multiple chemical potential. It would be interesting to find out whether numbers  $8\pi^2, 32\pi^2$  and  $2\pi^2$ , can be fixed in terms of central charges of dual gauge theories [15]. This will give us important information about universal behavior of dual gauge theories. It was shown in [12], that conductivity in the case of vanishing chemical potential can be determined exactly in terms of central charges. It is desirable to investigate whether electrical conductivity can also be fixed in terms of central charges in the presence of chemical potentials.
- It would be interesting to study  $\kappa_T$  and  $\frac{\kappa_T(\mu_1^2 + \mu_2^2 + \mu_3^2)}{\eta T}$  at extremality. Here one needs to be extremely careful about properly defining various quantities. But naively if we apply results obtained at extremality (take for example  $AdS_4$  Reissner-Nordström background, see [8] for details)

$$\eta \sim \mu^2, \quad \sigma \sim \left( \frac{\omega}{\mu} \right)^2. \quad (9.3)$$

Then taking<sup>16</sup>  $\epsilon + P = \rho\mu$ , we obtain

$$\kappa_T \sim \frac{\omega^2}{T}, \quad \frac{\kappa_T \mu^2}{\eta T} \sim \left( \frac{\omega}{T} \right)^2. \quad (9.4)$$

---

<sup>16</sup> $\epsilon = 4 \frac{r_0^3}{k_4^2 L^4}, \quad P = \frac{\epsilon}{2}, \quad \mu = \frac{\sqrt{3} r_0}{L^2}, \quad \rho = 2\sqrt{3} \frac{r_0^2}{k_4^2 L^2}$ , see [8].

# Acknowledgments

It is a pleasure to thank Somen Bhattacharjee, Sudipta Mukherji, Balram Rai, Sayan Chakrabarti, Anirban Basu, Sankhadeep Chakraborty, Binata Panda, Ambresh Shivaji, Souvik Banerjee for useful discussions. Special thanks to Balram Rai for discussion on chemical potential and to Sayan Chakrabarti, Sankhadeep Chakraborty, Binata Panda and in particular Sudipta Mukherji about improving the manuscript. Author is also thankful to referee for pointing out mistakes in the form of Schrödinger potential. Author would like to mention that he misses Prof. Alok Kumar and his constant encouraging words.

## A Appendix

Here we collect all the geometrical as well as thermodynamical quantities required for computation done in the text (see [2], [5], [6], [16] for details).

### A.1 Five dimensional black hole

In this subsection, we collect all the relevant information's for five dimensional R-charged black hole. Lagrangian is

$$\frac{\mathcal{L}}{\sqrt{-g}} = R - \frac{1}{4}G_{ij}F_{\mu\nu}^i F^{\mu\nu j} + \dots \quad (\text{A.1})$$

Where

$$G_{ij} = \frac{L^2}{2} \text{diag} [(X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2}] .$$

Metric and gauge fields are

$$ds_5^2 = -\mathcal{H}^{-2/3} \frac{(\pi T_0 L)^2}{u} f dt^2 + \mathcal{H}^{1/3} \frac{(\pi T_0 L)^2}{u} (dx^2 + dy^2 + dz^2) + \mathcal{H}^{1/3} \frac{L^2}{4fu^2} du^2, \quad (\text{A.2})$$

$$f(u) = \mathcal{H}(u) - u^2 \prod_{i=1}^3 (1 + \kappa_i), \quad H_i = 1 + \kappa_i u, \quad \mathcal{H} = \prod_{i=1}^3 H_i \quad (\text{A.3})$$

$$X^i = \frac{\mathcal{H}^{1/3}}{H_i(u)}, \quad A_t^i = \frac{\pi T_0 \sqrt{2k_i(1+k_1)(1+k_2)(1+k_3)u}}{H_i(u)}, \quad (\text{A.4})$$

Viscosity and various thermodynamical quantities are given by

$$T_H = \frac{2 + \kappa_1 + \kappa_2 + \kappa_3 - \kappa_1 \kappa_2 \kappa_3}{2\sqrt{(1+\kappa_1)(1+\kappa_2)(1+\kappa_3)}} T_0. \quad (\text{A.5})$$

$$s = \frac{\pi^2 N^2 T_0^3}{2} \prod_{i=1}^3 (1 + \kappa_i)^{1/2}, \quad (\text{A.6})$$

$$\eta = \frac{\pi N^2 T_0^3}{8} \prod_{i=1}^3 (1 + \kappa_i)^{1/2}, \quad (\text{A.7})$$

$$\varepsilon = \frac{3\pi^2 N^2 T_0^4}{8} \prod_{i=1}^3 (1 + \kappa_i), \quad (\text{A.8})$$

$$P = \frac{\pi^2 N^2 T_0^4}{8} \prod_{i=1}^3 (1 + \kappa_i). \quad (\text{A.9})$$

The densities of physical charges are

$$\rho_i = \frac{\pi N^2 T_0^3}{8} \sqrt{2\kappa_i} \prod_{l=1}^3 (1 + \kappa_l)^{1/2}. \quad (\text{A.10})$$

The chemical potentials conjugated to  $\rho_i$  are defined as

$$\mu_i = A_t^i(u) \Big|_{u=1} = \frac{\pi T_0 \sqrt{2\kappa_i}}{(1 + \kappa_i)} \prod_{l=1}^3 (1 + \kappa_l)^{1/2}. \quad (\text{A.11})$$

and

$$\frac{1}{16\pi G_5} = \frac{N_c^2}{16\pi^2 L^3}. \quad (\text{A.12})$$

## A.2 Four dimensional black hole

Metric and gauge fields in this case are

$$ds_4^2 = \frac{16(\pi T_0 L)^2}{9u^2} \mathcal{H}^{1/2} \left( -\frac{f}{\mathcal{H}} dt^2 + dx^2 + dz^2 \right) + \frac{L^2}{f u^2} \mathcal{H}^{1/2} du^2, \quad (\text{A.13a})$$

$$A_t^i = \frac{4}{3} \pi T_0 \sqrt{2\kappa_i \prod_{i=1}^4 (1 + \kappa_i)} \frac{u}{H_i}, \quad (\text{A.13b})$$

$$H_i = 1 + \kappa_i u, \quad (\text{A.13c})$$

$$\mathcal{H} = \prod_{i=1}^4 H_i, \quad (\text{A.13d})$$

$$f = \mathcal{H} - \prod_{i=1}^4 (1 + \kappa_i) u^3, \quad (\text{A.13e})$$

Thermodynamic quantities are given by

$$\epsilon = \sqrt{2} \pi^2 \left( \frac{2}{3} \right)^4 N_c^{3/2} T_0^3 \prod_{i=1}^4 (1 + \kappa_i) , \quad P = \frac{\sqrt{2} \pi^2}{3} \left( \frac{2}{3} \right)^3 N_c^{3/2} T_0^3 \prod_{i=1}^4 (1 + \kappa_i) , \quad (\text{A.14a})$$

$$s = \sqrt{2} \pi^2 \left( \frac{2}{3} \right)^3 N_c^{3/2} T_0^2 \prod_{i=1}^4 \sqrt{1 + \kappa_i} , \quad T = \frac{T_0 \left( 3 + \sum_{j=1}^4 k_i + \sum_{j>i, i,j=1}^4 k_i k_j - \prod_{i=1}^4 k_i \right)}{3 \sqrt{\prod_{i=1}^4 (1 + \kappa_i)}} \quad (\text{A.14b})$$

$$\rho_i = \sqrt{2} \pi \left( \frac{1}{3} \right)^3 N_c^{3/2} T_0^2 \sqrt{2 k_i \prod_{j=1}^4 (1 + \kappa_j)} , \quad \mu_i = \frac{4\pi T_0}{3} \frac{1}{1 + k_i} \sqrt{2 \kappa_i \prod_{i=1}^4 (1 + \kappa_i)} , \quad (\text{A.14c})$$

Other relevant expressions are

$$\eta = \frac{1}{4} \sqrt{2} \pi \left( \frac{2}{3} \right)^3 N_c^{3/2} T_0^2 \prod_{i=1}^4 \sqrt{1 + \kappa_i} \quad (\text{A.15})$$

$$G_{ij} = \frac{L^2}{2} \text{diag} [(X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2}, (X^4)^{-2}] .$$

$$X^i = \frac{\mathcal{H}^{1/4}}{H_i(u)} \quad (\text{A.16})$$

$$\frac{1}{16\pi G_4} = \frac{N_c^{\frac{3}{2}}}{24\sqrt{2}L^2} . \quad (\text{A.17})$$

### A.3 Seven dimensional black hole

$$ds_7^2 = \frac{4(\pi T_0 L)^2}{9u} \mathcal{H}^{1/5} \left( -\frac{f}{\mathcal{H}} dt^2 + dx_1^2 + \dots + dx_4^2 + dz^2 \right) + \frac{L^2}{4fu^2} \mathcal{H}^{1/5} du^2 \quad (\text{A.18a})$$

$$A_t = \frac{2}{3} \pi T_0 \sqrt{2\kappa_i \prod_{i=1}^2 (1 + \kappa_i)} \frac{u^2}{H_i}, \quad (\text{A.18b})$$

$$H_i = 1 + \kappa_i u^2, \quad (\text{A.18c})$$

$$\mathcal{H} = \prod_{i=1}^2 H_i, \quad (\text{A.18d})$$

$$f = \mathcal{H} - \prod_{i=1}^2 (1 + \kappa_i) u^3, \quad (\text{A.18e})$$

Thermodynamic quantities are given by

$$\epsilon = \frac{5\pi^3}{2} \left(\frac{2}{3}\right)^7 N_c^3 T_0^6 \prod_{i=1}^2 (1 + \kappa_i), \quad P = \frac{\pi^3}{2} \left(\frac{2}{3}\right)^7 N_c^3 T_0^6 \prod_{i=1}^2 (1 + \kappa_i), \quad (\text{A.19a})$$

$$s = 3\pi^3 \left(\frac{2}{3}\right)^7 N_c^3 T_0^5 \sqrt{\prod_{i=1}^2 (1 + \kappa_i)}, \quad T = \frac{T_0 (3 + \kappa_1 + \kappa_2 - \kappa_1 \kappa_2)}{3 \sqrt{\prod_{i=1}^2 (1 + \kappa_i)}}, \quad (\text{A.19b})$$

$$\rho_i = \pi^2 \left(\frac{2}{3}\right)^6 N_c^3 T_0^5 \sqrt{2\kappa_i \prod_{i=1}^2 (1 + \kappa_i)}, \quad \mu_i = \frac{2\pi T_0}{3(1 + \kappa_i)} \sqrt{2\kappa_i \prod_{i=1}^2 (1 + \kappa_i)}, \quad (\text{A.19c})$$

Other relevant results are

$$\eta = \frac{3}{4} \pi^2 \left(\frac{2}{3}\right)^7 N_c^3 T_0^5 \sqrt{\prod_{i=1}^2 (1 + \kappa_i)} \quad (\text{A.20})$$

$$G_{ij} = \frac{L^2}{2} \text{diag} [(X^1)^{-2}, (X^2)^{-2}].$$

$$X^i = \frac{\mathcal{H}^{2/5}}{H_i(u)} \quad (\text{A.21})$$

$$\frac{1}{16\pi G_7} = \frac{N_c^3}{6\pi^3 L^5}, \quad (\text{A.22})$$

## B Blackholes at extremality

Above black holes at extremality was constructed in [17]. Take

$$\bar{g}_{tt} = -f(u)A_1(u), \quad \bar{g}_{uu} = A_2(u)f^{-1}(u), \quad f(u) = (1-u)^2 V(u). \quad (\text{B.1})$$

Here we just give relevant information about  $f$ .

Dimension	Extremality condition	$V(u)$
5	$2 + \kappa_1 + \kappa_2 + \kappa_3 - \kappa_1\kappa_2\kappa_3 = 0$	$(1 + \kappa_1\kappa_2\kappa_3u)$
4	$3 + \sum_{j=1}^4 k_i + \sum_{i<j, i,j=1}^4 k_ik_j - \prod_{i=1}^4 k_i = 0$	$(1 + (2 + \sum_{j=1}^4 k_i)u + \prod_{i=1}^4 k_iu^2)$
7	$3 + \kappa_1 + \kappa_2 - \kappa_1\kappa_2 = 0$	$(1 + 2u + \kappa_1\kappa_2u^2)$

## References

- [1] D. T. Son and A. O. Starinets, JHEP **0603**, 052 (2006) [arXiv:hep-th/0601157].
- [2] K. Behrndt, M. Cvetič and W. A. Sabra, Nucl. Phys. B **553**, 317 (1999) [arXiv:hep-th/9810227]. M. Cvetič *et al.*, Nucl. Phys. B **558**, 96 (1999) [arXiv:hep-th/9903214].
- [3] J. Mas, “Shear viscosity from R-charged AdS black holes,” arXiv:hep-th/0601144. K. Maeda, M. Natsuume and T. Okamura, “Viscosity of gauge theory plasma with a chemical potential from AdS/CFT,” arXiv:hep-th/0602010. O. Saremi, “The Viscosity Bound Conjecture and Hydrodynamics of M2-Brane Theory at Finite Chemical Potential,” arXiv:hep-th/0601159. X. H. Ge, Y. Matsuo, F. W. Shu, S. J. Sin and T. Tsukioka, “Density Dependence of Transport Coefficients from Holographic Hydrodynamics,” Prog. Theor. Phys. **120**, 833 (2008) [arXiv:0806.4460 [hep-th]]; N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam and P. Surowka, “Hydrodynamics from charged black branes,” arXiv:0809.2596 [hep-th]; J. Erdmenger, M. Haack, M. Kaminski and A. Yarom, “Fluid dynamics of R-charged black holes,” JHEP **0901**, 055 (2009) [arXiv:0809.2488 [hep-th]]; S. Bhattacharyya, S. Lahiri, R. Loganayagam and S. Minwalla, “Large rotating AdS black holes from fluid mechanics,” JHEP **0809**, 054 (2008) [arXiv:0708.1770 [hep-th]]. M. Torabian and H. U. Yee, “Holographic nonlinear hydrodynamics from AdS/CFT with multiple/non-Abelian symmetries,” JHEP **0908**, 020 (2009) [arXiv:0903.4894 [hep-th]]. C. P. Herzog, P. Kovtun, S. Sachdev and D. T. Son, Phys. Rev. D **75**, 085020 (2007) [arXiv:hep-th/0701036].
- [4] D. T. Son and A. O. Starinets, JHEP **0209**, 042 (2002) [arXiv:hep-th/0205051]. G. Policastro, D. T. Son and A. O. Starinets, “From AdS/CFT correspondence to hydrodynamics,” JHEP **0209**, 043 (2002) [arXiv:hep-th/0205052]. P. Kovtun, D. T. Son and A. O. Starinets, “Viscosity in strongly interacting quantum field theories from black hole physics,” Phys. Rev. Lett. **94**, 111601 (2005) [arXiv:hep-th/0405231]. A. Buchel, “On universality of stress-energy tensor correlation functions in supergravity,” Phys. Lett. B **609**, 392 (2005) [arXiv:hep-th/0408095]. C. P. Herzog, JHEP **0212**, 026 (2002) [arXiv:hep-th/0210126]. C. P. Herzog, J. Phys. A **42**,

- 343001 (2009) [arXiv:0904.1975 [hep-th]]. S. A. Hartnoll, Class. Quant. Grav. **26**, 224002 (2009) [arXiv:0903.3246 [hep-th]]. J. McGreevy, arXiv:0909.0518[hep-th].
- [5] S. Jain, S. Mukherji and S. Mukhopadhyay, “Notes on R-charged black holes near criticality and gauge theory,” arXiv:0906.5134 [hep-th], JHEP 0911:051,2009
- [6] K. Maeda, M. Natsuume and T. Okamura, Phys. Rev. D **78**, 106007 (2008) [arXiv:0809.4074 [hep-th]].
- [7] T. Faulkner, H. Liu, J. McGreevy and D. Vegh, arXiv:0907.2694 [hep-th].
- [8] M. Edalati, J. I. Jottar and R. G. Leigh, arXiv:0910.0645 [hep-th]. E. Imeroni and A. Sinha, JHEP **0909**, 096 (2009) arXiv:0907.1892 [hep-th]. M. F. Paulos, arXiv:0910.4602[hep-th]. R. G. Cai, Y. Liu and Y-W. Sun, arXiv:0910.4705[hep-th]. S. K. Chakrabarti, S. Jain and S. Mukherji, arXiv:0910.5132 [hep-th]. D. Maity, S. Sarkar, N. Sircar, B. Sathiapalan and R. Shankar, arXiv:0909.4051 [hep-th]. B. Sahoo and H. U. Yee, arXiv:0910.5915 [hep-th]. N. Banerjee and S. Dutta, arXiv:0911.0557 [hep-th]. D. W. Pang, arXiv:0911.2777 [hep-th].
- [9] R. C. Myers, M. F. Paulos and A. Sinha, JHEP **0906**, 006 (2009) [arXiv:0903.2834 [hep-th]].
- [10] N. Iqbal and H. Liu, Phys. Rev. D **79**, 025023 (2009) arXiv:0809.3808 [hep-th].
- [11] P. Kovtun, D. T. Son and A. O. Starinets, JHEP **0310**, 064 (2003) [arXiv:hep-th/0309213].
- [12] P. Kovtun and A. Ritz, Phys. Rev. D **78**, 066009 (2008) [arXiv:0806.0110 [hep-th]].
- [13] M. Vojta, Y. Zhang, and S. Sachdev, Intl. Jour. of Mod. Phys. B **14**, 3719 (2000)., Markus Mueller and Lars Fritz and Subir Sachdev and Joerg Schmalian [arXiv.org:0910.5680]
- [14] G. T. Horowitz and M. M. Roberts, JHEP **0911**, 015 (2009) [arXiv:0908.3677 [hep-th]]. S. A. Hartnoll, C. P. Herzog and G. T. Horowitz, JHEP **0812**, 015 (2008) [arXiv:0810.1563 [hep-th]]. K. Goldstein, S. Kachru, S. Prakash and S. P. Trivedi, arXiv:0911.3586 [hep-th]. R. A. Konoplya and A. Zhidenko, arXiv:0909.2138 [hep-th].
- [15] S. Jain, arXiv:0912.2719 [hep-th].
- [16] S. S. Gubser, “Thermodynamics of spinning D3-branes,” Nucl. Phys. B **551**, 667 (1999) [arXiv:hep-th/9810225]. A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, “Charged AdS black holes and catastrophic holography,” Phys. Rev. D **60**, 064018 (1999) [arXiv:hep-th/9902170]. M. Cvetič and S. S. Gubser, “Phases of R-charged black holes, spinning branes and strongly coupled gauge theories,” JHEP

- 9904**, 024 (1999) [arXiv:hep-th/9902195]. R. G. Cai and K. S. Soh, “Critical behavior in the rotating D-branes,” *Mod. Phys. Lett. A* **14**, 1895 (1999) [arXiv:hep-th/9812121]. S. S. Gubser and I. Mitra, “Instability of charged black holes in anti-de Sitter space,” arXiv:hep-th/0009126.
- [17] H. Lu, J. w. Mei, C. N. Pope and J. F. Vazquez-Poritz, *Phys. Lett. B* **673**, 77 (2009) arXiv:0901.1677 [hep-th].